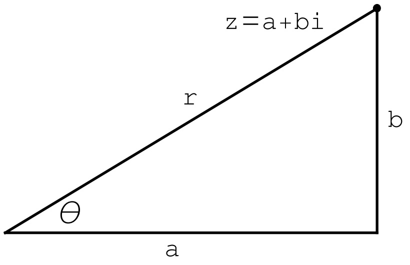
[10-02-01-T11] (Rev1)

Review trigonometry: complex numbers

■ Rectangular (form) and polar (trigonometric) form

Rectangular form of complex number: $a + b\,i$, where $a,\ b \in \mathbb{R}$. a is called the real part, b the imaginary part. The numbers $a + b\,i$ and $a - b\,i$ are conjugates and the conjugate of z is often denoted by $\overline{z}$ and sometimes by z^*. The product of conjugates is a real number, specifically $(a + b\,i)(a - b\,i) = a^2 + b^2$.

Polar form of complex number: $r(\cos\theta + i\sin\theta)$, where r is a real number called the *modulus* and θ is a real number called the *argument*. When $z = r(\cos\theta + i\sin\theta)$ is graphed in the complex plane, such a graph is called and *argand diagram*, r is the distance of the number z to the pole and θ is the angle of a ray from the pole to z with respect to the positive real axis, counterclockwise positive.



The expression $(\cos\theta + i\sin\theta)$ is often abbreviated to cis θ. It is a fact that $(\cos\theta + i\sin\theta) = e^{\theta i}$, where e is the base of the natural logarithm and $\theta \in \mathbb{R}$; $(\cos\theta + i\sin\theta) = e^{\theta i}$ is called Euler's formula.

■ Rectangular (form) ⟶ polar (trigonometric) form

[EX1] Write $z = 1 + i\sqrt{3}$ in polar form.

$$r = |z| = \sqrt{a^2 + b^2} = \sqrt{1 + \left(\sqrt{3}\right)^2} = \sqrt{1 + 3} = 2.$$

First note that $1 + i\sqrt{3}$ is in the first quadrant, then $\tan\theta = \frac{b}{a} = \frac{\sqrt{3}}{1} \Longrightarrow \theta = \frac{\pi}{3}$, or 60 °. Therefore, $1 + i\sqrt{3}$ is written in polar form as $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ or, if you prefer, $2\,\text{cis}\,\frac{\pi}{3}$. Equivalently, one may write $2\,e^{\frac{\pi}{3}i}$. □

[EX2] Write $z = -2 + 2\,i$ in polar form.

$r = |z| = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.

First note that $-2 + 2\,i$ is in the second quadrant, then $\tan\theta = \frac{b}{a} = -\frac{2}{2} \Longrightarrow \theta = \frac{3\pi}{4}$, or 135 °. Therefore, $-2 + 2\,i$ is written in polar form as $2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ or, if you prefer, $2\sqrt{2}\,\operatorname{cis}\frac{3\pi}{4}$. Equivalently, one may write $2\sqrt{2}\,e^{\frac{3\pi}{4}i}$. □

[EX3] Write $z = -2\,i$ in polar form.

Numbers on the axes are especially easy to rewrite in polar form. The modulus of $-2\,i$ is obviously 2 and if you picture the graph, you will quickly realize the argument is $\frac{3\pi}{2}$. Therefore, $-2\,\iota = 2\operatorname{cis}\frac{3\pi}{2}$. □

[EX4] Write $z = -5$ in polar form is $-5 = 5\operatorname{cis}\pi$. □

■ Polar form ⟶ rectangular form

A glance at the figure at the beginning of this discussion should convince you that $a = r\cos\theta$ and $b = r\sin\theta$.

[EX5] Write $z = 3\operatorname{cis}\frac{\pi}{3}$ in rectangular form.

Here, $a = 3\cos\frac{\pi}{3} = 3\left(\frac{1}{2}\right) = \frac{3}{2}$ and $b = 3\sin\frac{\pi}{3} = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$. Therefore, $3\operatorname{cis}\frac{\pi}{3} = \left(\frac{3}{2} + i\,\frac{3\sqrt{3}}{2}\right)$. □

[EX6] Write $z = 5\,e^{\frac{5\pi}{6}i}$ in rectangular form.

$a = 5\cos\frac{5\pi}{6} = \frac{-5\sqrt{3}}{2}$, $b = 5\sin\frac{5\pi}{6} = \frac{5}{2}$. So, $z = \frac{-5\sqrt{3}}{2} + \frac{5\,i}{2}$. □

[EX7] Write $z = 7\,e^{\pi i}$ in rectangular form.

A little thought (of the graph) helps us realize that $z = -7$. You could work this as in the previous example. □

■ Rectangular form ⟷ polar form in applied work

Although textbooks and teachers favor computations that involve only arguments that result in easily known exact values of the trigonometric functions, most practical applications do not involve these special arguments.

[EX8] Write $z = 3 + 2\,i$ in polar form.

The modulus is $\sqrt{13}$. For the argument, $\tan\theta = \frac{2}{3}$. Using tables or a machine, we find that $\theta = 33.6901\,°$. This is 0.5880 radians, if radians are desired; usually applied work is in degrees. Thus, $3 + 2\,i = \sqrt{13}\ \text{cis}\,(33.6901)$. □

[EX9] Write $z = 1.5\,e^{126\,°i}$ in rectangular form.

Then $a = 1.5\cos(126\,°) = 1.5\,(-0.5878) = -0.8817$. $b = 1.5\sin(126\,°) = 1.5\,(.8090) = 1.2135$. Therefore $z = -0.8817 + 1.2135\,i$. □

■ Products

The product of numbers in rectangular form can be found by multiplying binomials and replacing i^2 in the expansion with -1. The product of complex numbers in polar form is easily computed by the following formula: $(r_1\,\text{cis}\,\theta)\,(r_2\,\text{cis}\,\phi) = r_1\,r_2\,\text{cis}(\theta + \phi)$.

[EX10] $(1 + 2\,i)\,(3 - 5\,i) = 3 + 6\,i - 5\,i - 10\,i^2 = 13 + i$. □

[EX11] $\left(2\,\text{cis}\,\frac{\pi}{3}\right)\left(5\,\text{cis}\,\frac{2\pi}{4}\right) = 10\,\text{cis}\left(\frac{\pi}{3} + \frac{2\pi}{4}\right) = 10\,\text{cis}\left(\frac{5\pi}{6}\right)$. □

■ Quotients

The quotient of numbers in rectangular form can be found by multiplying by 1 in the form $\frac{\bar{z}}{\bar{z}}$. The quotient of numbers in polar form is exactly what you would expect having seen the product formula: $\frac{r_1\,\text{cis}\,\theta}{r_2\,\text{cis}\,\phi} = \frac{r_1}{r_2}\,\text{cis}\,(\theta - \phi)$.

[EX12] $(1 + 2\,i) \div (3 + 5\,i) = \frac{(1+2\,i)}{(3+5\,i)}\,\frac{(3-5\,i)}{(3-5\,i)} = \frac{(1+2\,i)\,(3-5\,i)}{(3+5\,i)\,(3-5\,i)} = \frac{13+i}{34}$. □

[EX13] $\left(2\,\text{cis}\,\frac{\pi}{3}\right) \div \left(5\,\text{cis}\,\frac{2\pi}{4}\right) = \frac{2}{5}\,\text{cis}\left(\frac{\pi}{3} - \frac{2\pi}{4}\right) = \frac{2}{5}\,\text{cis}\left(-\frac{\pi}{6}\right) = \frac{2}{5}\,\text{cis}\left(\frac{11\pi}{6}\right)$. □

■ Exponentiation and de Moivre's theorem

Exponentiation to an integer can be treated as repeated multiplication. This would not be a bad way at all to compute $(3+2i)^2 = (3+2i)(3+2i) = 5+12i$. But $(3+2i)^5$ would be laborious. De Moivre's theorem says that $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$. Written using Euler's formula, this becomes $\left[r\,e^{\theta i}\right]^n = r^n\,e^{n\theta i}$. In lower level courses this theorem is proved for $n \in \mathbb{Z}^+$, but is in fact valid for $n \in \mathbb{R}$. Euler's form of the complex number is especially handy, because all the rules you know for exponents apply.

[EX14] Compute $\left(2\operatorname{cis}\frac{\pi}{4}\right)^5$. Answer: $2^5\operatorname{cis}\frac{5\pi}{4} = 32\operatorname{cis}\frac{5\pi}{4}$. □

[EX15] Compute $\left(2\,e^{60°i}\right)^5$. Answer: $32\,e^{300°i}$. □

[EX16] Compute $\left(1+i\sqrt{3}\right)^{10}$. □

You could expand this using the binomial theorem, then replace all instances of i^2 by -1, and finally simplify. But, it is probably easier to rewrite $\left(1+i\sqrt{3}\right)$ in polar form and use de Moivre's theorem. Thus, $\left(1+i\sqrt{3}\right)^{10} = \left(2\operatorname{cis}\frac{\pi}{3}\right)^{10} = 2^{10}\operatorname{cis}\left(\frac{10\pi}{3}\right)$. If you prefer Euler's form,

$\left(1+i\sqrt{3}\right)^{10} = \left(2\,e^{\frac{\pi}{3}i}\right)^{10} = 2^{10}\,e^{\frac{10\pi}{3}i}$.

■ Extraction of roots, n^{th} root theorem

Perhaps the most important feature of the complex numbers is that every number possess as many roots as you desire. This means that a polynomial equation degree n has n distinct solutions. This fact is important enough to be called the Fundamental Theorem of Algebra. The following theorem tells how to find roots of a complex number.

Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. The z has n distinct nth roots

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \text{ where } k = 0, 1, 2, 3, \cdots, n-1.$$

Written using Euler's form of a complex number

$$w_k = r^{\frac{1}{n}}\, e^{\left[\frac{\theta+2k\pi}{n}\right]i}, \text{ where } k = 0, 1, 2, 3, \cdots, n-1.$$

[EX17] Find the four fourth roots of 4. Now, $4 = 4(\cos 0 + i\sin 0)$. So,

$$w_0 = 4^{\frac{1}{4}}\left[\cos\left(\frac{0+0}{4}\right) + i\sin\left(\frac{0+0}{4}\right)\right] = 4^{\frac{1}{4}}[\cos 0 + i\sin(0)] = 4^{\frac{1}{4}}$$

$$w_1 = 4^{\frac{1}{4}}\left[\cos\left(\frac{0+2\pi}{4}\right) + i\sin\left(\frac{0+2\pi}{4}\right)\right] = 4^{\frac{1}{4}}\left[\cos\frac{\pi}{2} + i\sin\left(\frac{\pi}{2}\right)\right] = 4^{\frac{1}{4}}\, i$$

$$w_2 = 4^{\frac{1}{4}}\left[\cos\left(\frac{0+4\pi}{4}\right) + i\sin\left(\frac{0+4\pi}{4}\right)\right] = 4^{\frac{1}{4}}[\cos\pi + i\sin(\pi)] = -4^{\frac{1}{4}}$$

$$w_3 = 4^{\frac{1}{4}}\left[\cos\left(\frac{0+6\pi}{4}\right) + i\sin\left(\frac{0+6\pi}{4}\right)\right] = 4^{\frac{1}{4}}\left[\cos\frac{3\pi}{2} + i\sin\left(\frac{3\pi}{2}\right)\right] = -i\,4^{\frac{1}{4}}$$

[EX18] Solve $x^3 + 8 = 0$. This amounts to finding the three third roots of -8. The argument may be radians or degrees. We'll use degrees for variety. $-8 = 8(\cos 180 + i\sin 0)$. So

$$w_0 = 2\left[\cos\left(\frac{180+0}{3}\right) + i\sin\left(\frac{180+0}{3}\right)\right] = 2[\cos 60 + i\sin(60)] = 2\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$$

$$w_1 = 2\left[\cos\left(\frac{180+360}{3}\right) + i\sin\left(\frac{180+360}{4}\right)\right] = 2[\cos 180 + i\sin(180)] = 2(-1 + 0\,i) = -2$$

$$w_2 = 2\left[\cos\left(\frac{180+720}{3}\right) + i\sin\left(\frac{180+720}{3}\right)\right] = 2[\cos 300 + i\sin(300)] = 2\left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$$